SYMMETRY THEOREMS FOR EXT VANISHING

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ABSTRACT. It was proved by Avramov and Buchweitz that if A is a commutative local complete intersection ring with finitely generated modules M and N, then the Ext groups between M and N vanish from some step if and only if the Ext groups between N and M vanish from some step.

This paper shows that the same is true under the weaker conditions that A is Gorenstein and that M and N have finite complete intersection dimension. The result is also proved if A is Gorenstein and has finite Cohen-Macaulay type.

Similar results are given for two types of non-commutative rings: Frobenius algebras and complete semi-local algebras.

0. INTRODUCTION

Let A be a commutative local complete intersection ring with finitely generated modules M and N. It is a surprising result of [3] that symmetry of Ext vanishing holds in the sense that

$$\operatorname{Ext}_A^i(M, N) = 0 \text{ for } i \gg 0 \ \Leftrightarrow \ \operatorname{Ext}_A^i(N, M) = 0 \text{ for } i \gg 0. \quad (1)$$

This paper proves that (1) remains true if A is a commutative local Gorenstein ring and M and N have finite complete intersection dimension in the sense of [4].

These conditions are weaker than the ones in [3] because a complete intersection ring is a Gorenstein ring for which each finitely generated module has finite complete intersection dimension.

It is also proved that (1) is true if A is a commutative local Gorenstein ring of finite Cohen-Macaulay type.

The method of the paper is to isolate two simple homological properties of (bi)modules, property (R) ("Rigidity") and property (S) ("Symmetry"), which make possible the abstract Ext vanishing result theorem 1.9. This in turn implies the results already stated, see theorem

4.1, and also two results dealing with non-commutative rings: Theorem 4.3 on certain Frobenius algebras and theorem 4.6 on certain complete semi-local algebras. In the non-commutative case, it turns out that M and N have to be A-bimodules, and symmetry of Ext vanishing takes the form

$$\mathrm{Ext}_A^i(M, N) = 0 \text{ for } i \gg 0 \ \Leftrightarrow \ \mathrm{Ext}_A^i(N, {}^\sigma M) = 0 \text{ for } i \gg 0$$

where σ is a "symmetrizing automorphism" of A.

Note that while the theorems of this paper are phrased using only classical homological algebra, some other parts of the paper use derived categories and functors such as RHom and $\overset{\mathrm{L}}{\otimes}$. However, the notation remains standard, and only standard properties of derived categories are used. Some background can be found in [13, sec. 2], and that paper also explains the ring theory notation which will be used.

The paper is organized as follows: After this introduction comes section 1 which introduces properties (R) and (S) and proves the abstract Ext vanishing result theorem 1.9. Next, sections 2 and 3 give some methods by which properties (R) and (S) can be established. And finally, section 4 uses the machinery to prove the concrete symmetry theorems for Ext vanishing 4.1, 4.3, and 4.6.

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1. An abstract Ext vanishing result

Definition 1.1 (Property (R))**.** Let A be a ring and let N be an A-left-module.

Suppose that if L is an exact complex of finitely generated projective A-right-modules for which the cohomology of $L \otimes_A N$ vanishes in high degrees, that is,

$$\mathrm{H}^{\gg 0}(L \otimes_A N) = 0,$$

then in fact, all the cohomology of $L \otimes_A N$ vanishes, that is,

$$\mathrm{H}(L \otimes_A N) = 0.$$

Then N is said to have property (R).

Remark 1.2. It is easy to see that if N has finite flat dimension, then it has property (R). But there are other modules with property (R), see section 2.

Lemma 1.3. *Let A be a noetherian ring with finite injective dimension from the left, $\mathrm{id}_A(A) < \infty$. Let M be a finitely generated A-left-module and let N be an A-bimodule.*

Suppose that N viewed as an A-left-module has property (R) and that $\mathrm{Ext}_A^{\gg 0}(M, N) = 0$. Then

$$\mathrm{RHom}_A(M, A \overset{\mathrm{L}}{\otimes}_A N) \cong \mathrm{RHom}_A(M, A) \overset{\mathrm{L}}{\otimes}_A N$$

in $\mathsf{D}(A^{\mathrm{op}})$, the derived category of A-right-modules.

Proof. Let

$$P = \cdots \to P^{-2} \to P^{-1} \to P^0 \to 0 \to \cdots$$

be a projective resolution of M consisting of finitely generated modules. Then P is a right-bounded complex of finitely generated projective A-left-modules. Hence $\mathrm{Hom}_A(P, A)$ is a left-bounded complex of finitely generated projective A-right-modules, and its cohomology is $\mathrm{Ext}_A(M, A)$ which is bounded because $\mathrm{id}_A(A) < \infty$. In consequence, there is a quasi-isomorphism $Q \xrightarrow{\simeq} \mathrm{Hom}_A(P, A)$ where Q is a right-bounded complex of finitely generated projective A-right-modules.

The quasi-isomorphism can be completed to a distinguished triangle

$$Q \xrightarrow{\simeq} \mathrm{Hom}_A(P, A) \longrightarrow L \longrightarrow,$$

where L is now an exact complex of finitely generated projective A-right-modules. This again gives a distinguished triangle

$$Q \otimes_A N \longrightarrow \mathrm{Hom}_A(P, A) \otimes_A N \longrightarrow L \otimes_A N \longrightarrow. \qquad (2)$$

Since Q and P are projective resolutions of $\mathrm{Hom}_A(P, A)$ and M, it follows that

$$Q \otimes_A N \cong \mathrm{Hom}_A(P, A) \overset{\mathrm{L}}{\otimes}_A N \cong \mathrm{RHom}_A(M, A) \overset{\mathrm{L}}{\otimes}_A N$$

and

$$\mathrm{Hom}_A(P, A) \otimes_A N \overset{\text{(a)}}{\cong} \mathrm{Hom}_A(P, A \otimes_A N) \cong \mathrm{RHom}_A(M, A \overset{\mathrm{L}}{\otimes}_A N),$$

where (a) is because each module in P is finitely generated projective. So the distinguished triangle (2) reads

$$\mathrm{RHom}_A(M, A) \overset{\mathrm{L}}{\otimes}_A N \longrightarrow \mathrm{RHom}_A(M, A \overset{\mathrm{L}}{\otimes}_A N) \longrightarrow L \otimes_A N \longrightarrow . \tag{3}$$

The cohomology of $\mathrm{RHom}_A(M, A)$ is $\mathrm{Ext}_A(M, A)$ which is bounded because $\mathrm{id}_A(A) < \infty$, as remarked above. Consequently, the cohomology of $\mathrm{RHom}_A(M, A) \overset{\mathrm{L}}{\otimes}_A N$ is right-bounded. And the cohomology of

$$\mathrm{RHom}_A(M, A \overset{\mathrm{L}}{\otimes}_A N) \cong \mathrm{RHom}_A(M, N)$$

is $\mathrm{Ext}_A(M, N)$ which is right-bounded by assumption.

The distinguished triangle (3) hence shows that the cohomology of $L \otimes_A N$ is right-bounded, that is, $\mathrm{H}^{\gg 0}(L \otimes_A N) = 0$. Since N viewed as an A-left-module has property (R), it follows that $\mathrm{H}(L \otimes_A N) = 0$, and hence the first morphism in (3),

$$\mathrm{RHom}_A(M, A) \overset{\mathrm{L}}{\otimes}_A N \longrightarrow \mathrm{RHom}_A(M, A \overset{\mathrm{L}}{\otimes}_A N),$$

is a quasi-isomorphism and hence an isomorphism in the derived category $\mathsf{D}(A^{\mathrm{op}})$. $\qquad\square$

Remark 1.4. Recall for the following setup that if σ is an automorphism of a ring A and M is an A-left-module, then there is an A-left-module $^\sigma M$ with scalar multiplication defined in terms of the scalar multiplication of M by $a \cdot m = \sigma(a)m$.

This procedure can also be applied to A-right-modules, and from either side to A-bimodules. Note that as A-bimodules,

$$^\sigma A \cong A^{\sigma^{-1}}.$$

Setup 1.5. Let A be a ring with an automorphism σ and suppose that the A-bimodule $^\sigma A$ has a resolution

$$I = \cdots \to 0 \to I^0 \to I^1 \to I^2 \to \cdots$$

of A-bimodules which are injective when viewed either as A-left-modules or A-right-modules.

Remark 1.6. If I is viewed as a complex of A-left-modules, then it is an injective resolution of A viewed as an A-left-module. Similarly from the right. Hence the functors

$$\mathrm{RHom}_A(-, {}^\sigma A) \quad \text{and} \quad \mathrm{RHom}_{A^{\mathrm{op}}}(-, {}^\sigma A)$$

can be defined as

$$\mathrm{Hom}_A(-, I) \quad \text{and} \quad \mathrm{Hom}_{A^{\mathrm{op}}}(-, I).$$

This has the advantage of giving RHom functors which are also defined on the derived category of A-bimodules, hence enabling the next definition.

Definition 1.7 (Property (S)). Let A and σ be as in setup 1.5 and let N be an A-bimodule which is finitely generated from either side.

Suppose that

$$\mathrm{RHom}_A(N, {}^\sigma A) \cong \mathrm{RHom}_{A^{\mathrm{op}}}(N, {}^\sigma A)$$

in $\mathsf{D}(A^{\mathrm{op}})$. Then N is said to have property (S).

Remark 1.8. It is clear that if A is commutative and σ is the identity, then an A-module N, viewed as an A-bimodule via $an = na$ for a in A and n in N, has property (S). But there are other modules with property (S), see section 3.

Theorem 1.9. *Let A and σ be as in setup 1.5 and suppose that A is noetherian and that $\mathrm{id}_A(A) < \infty$ and $\mathrm{id}_{A^{\mathrm{op}}}(A) < \infty$. Let M be a finitely generated A-left-module and let N be an A-bimodule which is finitely generated from either side.*

Suppose that N viewed as an A-left-module has property (R) and that N has property (S). Then

$$\mathrm{Ext}_A^{\gg 0}(M, N) = 0 \ \Rightarrow \ \mathrm{Ext}_A^{\gg 0}(N, {}^\sigma M) = 0.$$

Proof. I can compute,

$$\mathrm{RHom}_{A^{\mathrm{op}}}(\mathrm{RHom}_A(M, N), {}^\sigma A)$$

$$\cong \mathrm{RHom}_{A^{\mathrm{op}}}(\mathrm{RHom}_A(M, A \overset{\mathrm{L}}{\otimes}_A N), {}^\sigma A)$$

$$\overset{(\mathrm{r})}{\cong} \mathrm{RHom}_{A^{\mathrm{op}}}(\mathrm{RHom}_A(M, A) \overset{\mathrm{L}}{\otimes}_A N, {}^\sigma A)$$

$$\cong \mathrm{RHom}_{A^{\mathrm{op}}}(\mathrm{RHom}_A(M, A), \mathrm{RHom}_{A^{\mathrm{op}}}(N, {}^\sigma A))$$

$$\overset{(\mathrm{s})}{\cong} \mathrm{RHom}_{A^{\mathrm{op}}}(\mathrm{RHom}_A(M, A), \mathrm{RHom}_A(N, {}^\sigma A))$$

$$\cong \mathrm{RHom}_A(N, \mathrm{RHom}_{A^{\mathrm{op}}}(\mathrm{RHom}_A(M, A), {}^\sigma A))$$

$$\overset{(\mathrm{t})}{\cong} \mathrm{RHom}_A(N, \mathrm{RHom}_{A^{\mathrm{op}}}(A, {}^\sigma A) \overset{\mathrm{L}}{\otimes}_A M)$$

$$\cong \mathrm{RHom}_A(N, {}^\sigma A \overset{\mathrm{L}}{\otimes}_A M)$$

$$\cong \mathrm{RHom}_A(N, {}^\sigma M).$$

Here (r) is by lemma 1.3 since N viewed as an A-left-module has property (R), and (s) is since N has property (S), while (t) is because ${}^\sigma A$

viewed as an A-right-module is just A, so $\mathrm{id}_{A^{\mathrm{op}}}(^\sigma A) = \mathrm{id}_{A^{\mathrm{op}}}(A) < \infty$, and this implies

$$\mathrm{RHom}_{A^{\mathrm{op}}}(\mathrm{RHom}_A(M, A), {}^\sigma A) \cong \mathrm{RHom}_{A^{\mathrm{op}}}(A, {}^\sigma A) \overset{\mathrm{L}}{\otimes}_A M.$$

The remaining isomorphisms are standard.

Now, the condition

$$\mathrm{Ext}_A^{\gg 0}(M, N) = 0$$

says that the cohomology of

$$\mathrm{RHom}_A(M, N)$$

is bounded. But since $\mathrm{id}_{A^{\mathrm{op}}}(^\sigma A) < \infty$, this implies that the cohomology of

$$\mathrm{RHom}_{A^{\mathrm{op}}}(\mathrm{RHom}_A(M, N), {}^\sigma A)$$

is bounded. And then the above computation shows that the cohomology of

$$\mathrm{RHom}_A(N, {}^\sigma M)$$

is bounded, that is,

$$\mathrm{Ext}_A^{\gg 0}(N, {}^\sigma M) = 0.$$

$\square$

Remark 1.10. Definition 1.7 and theorem 1.9 would be simpler without the automorphism σ. However, it appears that to get a theory covering any reasonable stock of non-commutative rings, σ is a necessary ingredient. See remark 3.2 and proposition 3.9.

2. PROPERTY (R)

This section gives some methods by which property (R) from definition 1.1 can be established.

The following definition is classical.

Definition 2.1. Let A be a ring and let N be an A-left-module.

Suppose that there exists $c \geqslant 0$ so that

$$\mathrm{Tor}_i^A(Z, N) = \cdots = \mathrm{Tor}_{i+c}^A(Z, N) = 0 \ \Rightarrow \ \mathrm{Tor}_{\geqslant i}^A(Z, N) = 0$$

for each A-right-module Z and each $i \geqslant 1$. Then N is said to be Tor rigid.

Proposition 2.2. *Let A be a ring and let N be an A-left-module which is* Tor *rigid.*

Then N has property (R).

Proof. Let L be an exact complex of finitely generated projective A-right-modules with

$$H^{\gg 0}(L \otimes_A N) = 0.$$

Suspending L if necessary, I can suppose

$$H^{\geqslant -c-1}(L \otimes_A N) = 0 \qquad (4)$$

where c is the constant from definition 2.1.

Since L is exact,

$$\cdots \to L^{-2} \to L^{-1} \to L^0 \to Z \to 0$$

is a projective resolution of the A-right-module $Z = \operatorname{Coker}(L^{-1} \to L^0)$, so

$$\operatorname{Tor}_j^A(Z, N) \cong H^{-j}(L \otimes_A N) \qquad (5)$$

for $j \geqslant 1$.

Combining equations (4) and (5) shows

$$\operatorname{Tor}_1^A(Z, N) = \cdots = \operatorname{Tor}_{c+1}^A(Z, N) = 0,$$

and Tor rigidity of N now implies

$$\operatorname{Tor}_j^A(Z, N) = 0$$

for each $j \geqslant 1$. Combining with equation (5) shows

$$H^{\leqslant -1}(L \otimes_A N) = 0,$$

and combining with equation (4) shows

$$H(L \otimes_A N) = 0$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following definition was first made in the commutative case in [8, dfn. 3.1].

Definition 2.3 (the AB property)**.** Let A be a ring. Suppose that there exists $c \geqslant 1$ so that

$$\operatorname{Ext}_A^{\gg 0}(M, N) = 0 \implies \operatorname{Ext}_A^{\geqslant c}(M, N) = 0$$

when M and N are finitely generated A-left-modules.

Then A is said to have the left AB property.

Proposition 2.4. *Let A be a ring with* $\operatorname{id}_{A^{\mathrm{op}}}(A) < \infty$ *which has the left* AB *property.*

Then each finitely generated A-left-module N has property (R)*.*

Proof. The condition $\mathrm{id}_{A^{\mathrm{op}}}(A) < \infty$ implies that A viewed as an A-right-module has a bounded injective resolution,

$$I = \cdots \to 0 \to I^0 \to \cdots \to I^d \to 0 \to \cdots .$$

There is a quasi-isomorphism $A_A \xrightarrow{\simeq} I$ where A_A is A viewed as an A-right-module. Let L be an exact complex of finitely generated projective A-right-modules. Since L consists of projective modules, the functor $\mathrm{Hom}_{A^{\mathrm{op}}}(L, -)$ preserves quasi-isomorphisms of bounded complexes, so there is a quasi-isomorphism

$$\mathrm{Hom}_{A^{\mathrm{op}}}(L, A) \xrightarrow{\simeq} \mathrm{Hom}_{A^{\mathrm{op}}}(L, I).$$

The complex I is bounded and consists of injective modules, so the functor $\mathrm{Hom}_{A^{\mathrm{op}}}(-, I)$ preserves exactness, so $\mathrm{Hom}_{A^{\mathrm{op}}}(L, I)$ is exact. Hence

$$L^* = \mathrm{Hom}_{A^{\mathrm{op}}}(L, A)$$

is also exact.

Since L consists of finitely generated projective A-right-modules, there is an isomorphism $L \xrightarrow{\cong} \mathrm{Hom}_A(L^*, A)$. Hence, if N is a finitely generated A-left-module,

$$
\begin{aligned}
L \otimes_A N &\cong \mathrm{Hom}_A(L^*, A) \otimes_A N \\
&\stackrel{\mathrm{(a)}}{\cong} \mathrm{Hom}_A(L^*, A \otimes_A N) \\
&\cong \mathrm{Hom}_A(L^*, N),
\end{aligned}
\tag{6}
$$

where (a) is because L^* consists of finitely generated projective A-left-modules.

Since L^* is exact, for each i,

$$\cdots \to L^{*i-2} \to L^{*i-1} \to L^{*i} \to Z \to 0$$

is a projective resolution of the finitely generated A-left-module $Z = \mathrm{Coker}\,(L^{*i-1} \to L^{*i})$. Combining with equation (6) shows

$$\mathrm{H}^j(L \otimes_A N) \cong \mathrm{H}^j \mathrm{Hom}_A(L^*, N) \cong \mathrm{Ext}_A^{j+i}(Z, N) \tag{7}$$

for $j + i \geqslant 1$.

Now suppose

$$\mathrm{H}^{\gg 0}(L \otimes_A N) = 0.$$

Then equation (7) shows $\mathrm{Ext}_A^{\gg 0}(Z, N) = 0$, and the left AB property implies $\mathrm{Ext}_A^{\geqslant c}(Z, N) = 0$ which by equation (7) says

$$\mathrm{H}^j(L \otimes_A N) = 0$$

for $j + i \geqslant c$, that is, $j \geqslant c - i$. Varying i now shows

$$\mathrm{H}(L \otimes_A N) = 0$$

as desired. □

Proposition 2.5. *Let A be a left-noetherian ring with $\mathrm{id}_A(A) < \infty$ which has a finite set $\mathcal{C}$ of left-modules so that if N is a finitely generated A-left-module with $\mathrm{Ext}_A^{\geqslant 1}(N, A) = 0$, then N is isomorphic to the direct sum of finitely many modules from $\mathcal{C}$.*

Then A has the left AB property.

Proof. The set $\mathcal{C}$ is finite, so it is clear that there exists $\widetilde{c} \geqslant 1$ so that if $\widetilde{M}$ and $\widetilde{N}$ are isomorphic to direct sums of finitely many modules from $\mathcal{C}$, then

$$\mathrm{Ext}_A^{\gg 0}(\widetilde{M}, \widetilde{N}) = 0 \implies \mathrm{Ext}_A^{\geqslant \widetilde{c}}(\widetilde{M}, \widetilde{N}) = 0. \tag{8}$$

Let M be a finitely generated A-left-module and write $d = \mathrm{id}_A(A)$. Let

$$0 \to \widetilde{M} \to P_{d-1} \to \cdots \to P_0 \to M \to 0$$

be an exact sequence where the P_i are finitely generated projective A-left-modules. Since $\widetilde{M}$ is the d'th syzygy in a projective resolution of M,

$$\mathrm{Ext}_A^{d+i}(M, N) \cong \mathrm{Ext}_A^i(\widetilde{M}, N) \tag{9}$$

for each A-left-module N and each $i \geqslant 1$. If $N = Q$ is finitely generated projective, then this implies

$$\mathrm{Ext}_A^i(\widetilde{M}, Q) = 0 \tag{10}$$

for $i \geqslant 1$, since $\mathrm{id}_A(Q) \leqslant \mathrm{id}_A(A) = d$. In particular, $\mathrm{Ext}_A^{\geqslant 1}(\widetilde{M}, A) = 0$, so $\widetilde{M}$ is isomorphic to the direct sum of finitely many modules from $\mathcal{C}$.

Now let N be any finitely generated A-left-module, and let

$$0 \to \widetilde{N} \to Q_{d-1} \to \cdots \to Q_0 \to N \to 0$$

be an exact sequence where the Q_i are finitely generated projective A-left-modules. Like $\widetilde{M}$, the module $\widetilde{N}$ is isomorphic to the direct sum of finitely many modules from $\mathcal{C}$. Splitting the exact sequence into short exact sequences and applying the long exact sequence of Ext groups repeatedly along with equation (10) shows

$$\mathrm{Ext}_A^i(\widetilde{M}, N) \cong \mathrm{Ext}_A^{d+i}(\widetilde{M}, \widetilde{N})$$

for $i \geqslant 1$, and combining this with equation (9) shows

$$\mathrm{Ext}_A^j(M, N) \cong \mathrm{Ext}_A^j(\widetilde{M}, \widetilde{N}) \ \text{ for } \ j \geqslant d + 1. \tag{11}$$

Setting $c = \max\{\widetilde{c}, d + 1\}$ and combining equations (8) and (11) now shows

$$\operatorname{Ext}_A^{\gg 0}(M, N) = 0 \;\Rightarrow\; \operatorname{Ext}_A^{\geq c}(M, N) = 0$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. Property (S)

This section gives some methods by which property (S) from definition 1.7 can be established.

Remark 3.1. Let A be a commutative ring and let σ be the identity automorphism of A. This clearly gives the situation of setup 1.5.

If N is a finitely generated A-module, then I can view N as an A-bimodule via $an = na$ for a in A and n in N, and it is obvious that

$$\operatorname{RHom}_A(N, A) \cong \operatorname{RHom}_{A^{\mathrm{op}}}(N, A)$$

in $\mathsf{D}(A^{\mathrm{op}})$, so N has property (S).

Remark 3.2. Let A be a (finite dimensional) Frobenius algebra over the field k and let σ be a symmetrizing automorphism of A. This means that σ is an automorphism for which $\operatorname{Hom}_k(A, k) \cong {}^{\sigma}A$ as A-bimodules.

Observe that ${}^{\sigma}A$ is injective as an A-module from either side, so ${}^{\sigma}A$ is a resolution of itself by A-bimodules which are injective when viewed either as A-left-modules or as A-right-modules. Hence I am in the situation of setup 1.5.

Let N be an A-bimodule which is finitely generated from either side. Then

$$\operatorname{RHom}_A(N, {}^{\sigma}A) \cong \operatorname{RHom}_A(N, \operatorname{Hom}_k(A, k))$$

$$\cong \operatorname{Hom}_k(A \overset{\mathrm{L}}{\otimes}_A N, k)$$

$$\cong \operatorname{Hom}_k(N \overset{\mathrm{L}}{\otimes}_A A, k)$$

$$\cong \operatorname{RHom}_{A^{\mathrm{op}}}(N, \operatorname{Hom}_k(A, k))$$

$$\cong \operatorname{RHom}_{A^{\mathrm{op}}}(N, {}^{\sigma}A)$$

in $\mathsf{D}(A^{\mathrm{op}})$, so N has property (S).

Setup 3.3. Let k be a field and let A be a complete semi-local noetherian k-algebra.

Remark 3.4. The conditions of completeness and semi-locality in the setup mean that, if $\mathfrak{m}$ denotes the Jacobson radical of A, then A is complete in the $\mathfrak{m}$-adic topology while $A/\mathfrak{m}$ is semi-simple with $\dim_k A/\mathfrak{m} < \infty$.

Duality with respect to k will be denoted by

$$(-)' = \operatorname{Hom}_k(-, k).$$

This functor interchanges A-left-modules and A-right-modules, and sends A-bimodules to A-bimodules.

According to [7, lem. 2.5], the A-bimodule

$$E = \operatorname{colim}(A/\mathfrak{m}^n)'$$

is injective from either side and induces a Morita self-duality for A; see [1, §24] for background on Morita duality.

The Morita duality functors will be denoted by

$$(-)^L = \operatorname{Hom}_A(-, E), \quad (-)^R = \operatorname{Hom}_{A^{\mathrm{op}}}(-, E).$$

The functor $(-)^L$ sends A-left-modules to A-right-modules and A-bimodules to A-bimodules. It sends modules finitely generated from the left to modules artinian from the right, and modules artinian from the left to modules finitely generated from the right; see [1, prop. 10.10 and thms. 24.5 and 24.6]. If X is an A-left-module or an A-bimodule and X is either finitely generated or artinian from the left, then there is an isomorphism

$$(X^L)^R \cong X \tag{12}$$

by [1, prop. 10.10 and thm. 24.6]. Of course, this can all be dualized.

The functors $(-)'$, $(-)^L$, and $(-)^R$ are exact, and so remain well defined on derived categories.

Lemma 3.5. *Let A be as in setup 3.3 and let Y be an A-bimodule which is finitely generated from the right.*

Then there is an isomorphism of A-bimodules

$$(Y^R)' \cong Y.$$

Proof. First observe

$$
\begin{aligned}
Y^R &= \operatorname{Hom}_{A^{\mathrm{op}}}(Y, E) \\
&= \operatorname{Hom}_{A^{\mathrm{op}}}(Y, \operatorname{colim}(A/\mathfrak{m}^n)') \\
&\overset{\text{(a)}}{\cong} \operatorname{colim} \operatorname{Hom}_{A^{\mathrm{op}}}(Y, (A/\mathfrak{m}^n)') \\
&= \operatorname{colim} \operatorname{Hom}_{A^{\mathrm{op}}}(Y, \operatorname{Hom}_k(A/\mathfrak{m}^n, k)) \\
&\cong \operatorname{colim} \operatorname{Hom}_k(Y \otimes_A A/\mathfrak{m}^n, k) \\
&= \operatorname{colim}(Y \otimes_A A/\mathfrak{m}^n)' \\
&\cong \operatorname{colim}(Y/Y\mathfrak{m}^n)',
\end{aligned}
$$

where (a) is because Y is finitely generated from the right (this was re-marked already in [7, lem. 5.9(1)]). Observe also that since $\dim_k A/\mathfrak{m} < \infty$, it is more generally true that $\dim_k A/\mathfrak{m}^n < \infty$ for each $n \geqslant 1$. Since $Y/Y\mathfrak{m}^n$ is finitely generated from the right over $A/\mathfrak{m}^n$, it follows that $\dim_k Y/Y\mathfrak{m}^n < \infty$ for each $n \geqslant 1$.

These observations imply (b) and (c) in

$$
\begin{aligned}
(Y^R)' &= \operatorname{Hom}_k(Y^R, k) \\
&\overset{\text{(b)}}{\cong} \operatorname{Hom}_k(\operatorname{colim}(Y/Y\mathfrak{m}^n)', k) \\
&\cong \lim \operatorname{Hom}_k((Y/Y\mathfrak{m}^n)', k) \\
&\overset{\text{(c)}}{\cong} \lim Y/Y\mathfrak{m}^n \\
&= \widehat{Y} \\
&\overset{\text{(d)}}{\cong} Y \otimes_A \widehat{A} \\
&\overset{\text{(e)}}{\cong} Y,
\end{aligned}
$$

where $\widehat{Y}$ denotes the completion of Y in the $\mathfrak{m}$-adic topology from the right, where (d) is by [13, lem. 2.4(1)], and where (e) holds because $\widehat{A} \cong A$ by assumption. $\qquad\square$

Lemma 3.6. *Let A be as in setup 3.3 and let Z be an A-bimodule which is artinian from either side.*

Then there is an isomorphism of A-bimodules

$$Z^L \cong Z^R.$$

Proof. This is a computation,

$$Z^L \overset{\text{(a)}}{\cong} ((Z^L)^R)' \overset{\text{(b)}}{\cong} Z' \overset{\text{(c)}}{\cong} ((Z^R)^L)' \overset{\text{(d)}}{\cong} Z^R.$$

Here (a) holds by lemma 3.5 because Z^L is finitely generated from the right by remark 3.4, and (b) is by equation (12) because Z is artinian from the left. Similarly, (c) is by the dual of (12) because Z is artinian from the right, and (d) is by the dual of lemma 3.5 because Z^R is finitely generated from the left by remark 3.4. $\qquad\square$

Remark 3.7. The $\mathfrak{m}$-left-torsion functor over A is defined by

$$\Gamma_{\mathfrak{m}}(M) = \{\, m \in M \mid \mathfrak{m}^n m = 0 \text{ for } n \gg 0 \,\}.$$

This functor sends A-left-modules to A-left-modules and A-bimodules to A-bimodules. It has a derived functor $\mathrm{R}\Gamma_{\mathfrak{m}}$ defined on derived categories.

There is also an $\mathfrak{m}$-right-torsion functor $\Gamma_{\mathfrak{m}^{\mathrm{op}}}$ with derived functor $R\Gamma_{\mathfrak{m}^{\mathrm{op}}}$. See [13, sec. 6].

Lemma 3.8. *Let A be as in setup 3.3, suppose that A has a balanced dualizing complex (see [7, dfn. 3.7]), and let N be an A-bimodule which is finitely generated from either side.*

Then

$$\mathrm{RHom}_A(N, R\Gamma_{\mathfrak{m}}(A)^L) \cong \mathrm{RHom}_{A^{\mathrm{op}}}(N, R\Gamma_{\mathfrak{m}^{\mathrm{op}}}(A)^R)$$

in $\mathsf{D}(A \otimes_k A^{\mathrm{op}})$, the derived category of A-bimodules.

Proof. The cohomology modules of $R\Gamma_{\mathfrak{m}}(N)$ are artinian from either side by the proof of [7, thm. 3.5(3)], and so lemma 3.6 implies

$$R\Gamma_{\mathfrak{m}}(N)^L \cong R\Gamma_{\mathfrak{m}}(N)^R \tag{13}$$

in $\mathsf{D}(A \otimes_k A^{\mathrm{op}})$.

Since A has a balanced dualizing complex, it satisfies the χ condition, as defined in [7, p. 289], by [7, cor. 3.9]. Also, A is complete in the $\mathfrak{m}$-adic topology, so $\mathfrak{m}$ has the left and right Artin-Rees properties by [9, thm. 1.1]. Hence

$$R\Gamma_{\mathfrak{m}}(N) \cong R\Gamma_{\mathfrak{m}^{\mathrm{op}}}(N)$$

in $\mathsf{D}(A \otimes_k A^{\mathrm{op}})$ by [12, thm. 2.9], so

$$R\Gamma_{\mathfrak{m}}(N)^R \cong R\Gamma_{\mathfrak{m}^{\mathrm{op}}}(N)^R.$$

Combining with equation (13) gives

$$R\Gamma_{\mathfrak{m}}(N)^L \cong R\Gamma_{\mathfrak{m}^{\mathrm{op}}}(N)^R,$$

yielding (b) in

$$\mathrm{RHom}_A(N, R\Gamma_{\mathfrak{m}}(A)^L) \overset{\mathrm{(a)}}{\cong} R\Gamma_{\mathfrak{m}}(N)^L$$

$$\overset{\mathrm{(b)}}{\cong} R\Gamma_{\mathfrak{m}^{\mathrm{op}}}(N)^R$$

$$\overset{\mathrm{(c)}}{\cong} \mathrm{RHom}_{A^{\mathrm{op}}}(N, R\Gamma_{\mathfrak{m}^{\mathrm{op}}}(A)^R)$$

where (a) and (c) are by the version of local duality in [12, thm. 3.6(2)]. □

Proposition 3.9. *Let A be a complete semi-local noetherian k-algebra over the field k, which is Gorenstein in the sense that there is an automorphism σ so that the d'th suspension $\Sigma^d(^\sigma A)$ is a balanced dualizing complex (see [7, dfn. 3.7]).*

Then A and σ give the situation of setup 1.5, and each A-bimodule N which is finitely generated from either side has property (S).

Proof. To get the resolution I in setup 1.5, view the A-bimodule ${}^{\sigma}A$ as a left-module over the enveloping algebra $A \otimes_k A^{\mathrm{op}}$, take an injective resolution I, and view I as a complex of A-bimodules.

By the proof of [7, cor. 3.9], both $\mathrm{R\Gamma}_{\mathfrak{m}}(A)^L$ and $\mathrm{R\Gamma}_{\mathfrak{m}^{\mathrm{op}}}(A)^R$ are balanced dualizing complexes for A. But the balanced dualizing complex is unique by [7, p. 300], so

$$\mathrm{R\Gamma}_{\mathfrak{m}}(A)^L \cong \Sigma^d({}^{\sigma}A) \cong \mathrm{R\Gamma}_{\mathfrak{m}^{\mathrm{op}}}(A)^R.$$

Substituting this into lemma 3.8 gives

$$\mathrm{RHom}_A(N, \Sigma^d({}^{\sigma}A)) \cong \mathrm{RHom}_{A^{\mathrm{op}}}(N, \Sigma^d({}^{\sigma}A))$$

in $\mathsf{D}(A \otimes_k A^{\mathrm{op}})$, hence in particular in $\mathsf{D}(A^{\mathrm{op}})$. Using Σ^{-d} then proves property (S) for N. $\qquad\square$

4. Symmetry theorems for Ext vanishing

This section applies the theory of the previous sections to show symmetry theorems for Ext vanishing over commutative rings, Frobenius algebras, and (non-commutative) complete semi-local algebras.

The following theorem on commutative rings uses complete intersection dimension, CI-dim, as introduced in [4]. Note that a module over a commutative ring can be viewed either as a left-module, a right-module, or a bimodule via $an = na$ for a in A and n in N. In consequence, the left AB property will just be called the AB property.

Theorem 4.1. *Let A be a commutative local noetherian Gorenstein ring, and let M and N be finitely generated A-modules.*

Suppose either that

$$\mathrm{CI\text{-}dim}_A M < \infty \text{ and } \mathrm{CI\text{-}dim}_A N < \infty,$$

or that A has the AB property (see definition 2.3). Then

$$\mathrm{Ext}_A^{\gg 0}(M, N) = 0 \iff \mathrm{Ext}_A^{\gg 0}(N, M) = 0.$$

Moreover, if A has finite Cohen-Macaulay type, then it has the AB property.

Proof. The proof is an application of theorem 1.9, so I need to check the conditions of that theorem.

Let σ be the identity automorphism. This clearly gives the situation of setup 1.5.

The ring A is commutative, local, and Gorenstein, so $\mathrm{id}_A(A) < \infty$ and $\mathrm{id}_{A^{\mathrm{op}}}(A) < \infty$.

Remark 3.1 says that M and N, viewed as A-bimodules, have property (S).

If CI-dim$_A M < \infty$ and CI-dim$_A N < \infty$, then [10, cor. 2.3] implies that M and N are Tor rigid in the sense of definition 2.1, and then M and N, viewed as A-left-modules, have property (R) by proposition 2.2. If A has the AB property then M and N, viewed as A-left-modules, have property (R) by proposition 2.4.

So theorem 1.9 gives

$$\operatorname{Ext}_A^{\gg 0}(M, N) = 0 \;\Rightarrow\; \operatorname{Ext}_A^{\gg 0}(N, M) = 0,$$

and theorem 1.9 applied to N and M gives

$$\operatorname{Ext}_A^{\gg 0}(N, M) = 0 \;\Rightarrow\; \operatorname{Ext}_A^{\gg 0}(M, N) = 0.$$

This establishes the implications of the theorem.

Finally, a finitely generated A-module N with $\operatorname{Ext}_A^{\geq 1}(N, A) = 0$ is maximal Cohen-Macaulay by [5, thm. 3.3.7 and cor. 3.5.11], so such an N is a direct sum of finitely many indecomposable maximal Cohen-Macaulay modules. Thus, if A has finite Cohen-Macaulay type, the set $\mathcal{C}$ in proposition 2.5 can be taken to be a set of representatives of the finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay modules, and so in this case, A has the AB property. $\square$

Remark 4.2. The AB case of theorem 4.1 is known already from [8], but the CI-dim and finite Cohen-Macaulay type cases appear to be new.

Note that a local complete intersection ring has the AB property by [8, cor. 3.5], so there is a non-trivial supply of rings with the AB property. Note also that the CI-dim of any finitely generated module over a local complete intersection ring is finite by [4, thm. 1.3], so again, there is a non-trivial supply of modules with finite CI-dim.

Theorem 4.3. *Let A be a (finite dimensional) Frobenius algebra over the field k, let σ be a symmetrizing automorphism of A, that is, an automorphism for which $\operatorname{Hom}_k(A, k) \cong {}^\sigma A$ as A-bimodules, and let M and N be A-bimodules which are finitely generated from either side.*

Suppose that A has the left AB property. Then

$$\operatorname{Ext}_A^{\gg 0}(M, N) = 0 \;\Leftrightarrow\; \operatorname{Ext}_A^{\gg 0}(N, {}^\sigma M) = 0.$$

Moreover, if A has finite representation type, then it has the left AB property.

Proof. The proof is again an application of theorem 1.9.

The algebra A and the automorphism σ give the situation of setup 1.5 by remark 3.2.

The algebra A is Frobenius, so in particular self-injective from either side, so $\mathrm{id}_A(A) = 0 < \infty$ and $\mathrm{id}_{A^{\mathrm{op}}}(A) = 0 < \infty$.

Since A has the left AB property, N and ${}^\sigma M$ viewed as A-left-modules have property (R) by proposition 2.4.

The A-bimodules N and ${}^\sigma M$ have property (S) by remark 3.2.

So theorem 1.9 gives
$$\mathrm{Ext}_A^{\gg 0}(M, N) = 0 \ \Rightarrow\ \mathrm{Ext}_A^{\gg 0}(N, {}^\sigma M) = 0,$$
and theorem 1.9 applied to N and ${}^\sigma M$ gives
$$\mathrm{Ext}_A^{\gg 0}(N, {}^\sigma M) = 0 \ \Rightarrow\ \mathrm{Ext}_A^{\gg 0}({}^\sigma M, {}^\sigma N) = 0,$$
that is,
$$\mathrm{Ext}_A^{\gg 0}(N, {}^\sigma M) = 0 \ \Rightarrow\ \mathrm{Ext}_A^{\gg 0}(M, N) = 0.$$
This establishes the implications of the theorem.

Finally, if A has finite representation type, then proposition 2.5 clearly implies that A has the AB property. $\qquad\square$

Remark 4.4. There is a significant supply of Frobenius algebras of finite representation type to which theorem 4.3 applies, for instance, group algebras kG where k is a field of characteristic $p > 0$ and G is a finite group whose order is divisible by p and whose Sylow p-subgroups are cyclic, cf. [2, thm. VI.3.3]. Here the automorphism σ is even the identity since kG is a symmetric algebra.

The following definition was basically made already in [6, dfn. 8.5] and [6, dfn. 9.9].

Definition 4.5. Let A be a complete semi-local noetherian k-algebra over the field k for which the $\mathfrak{m}$-left-torsion functor $\Gamma_{\mathfrak{m}}$ from remark 3.7 has finite cohomological dimension d.

A finitely generated A-left-module M is called maximal Cohen-Macaulay if
$$\mathrm{R}^{\leqslant d-1}\Gamma_{\mathfrak{m}}(M) = 0.$$
If there are only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay A-left-modules, then A is said to have finite Cohen-Macaulay type.

Theorem 4.6. *Let A be a complete semi-local noetherian algebra over the field k, which is Gorenstein in the sense that there is an automorphism σ of A so that the d'th suspension $\Sigma^d({}^\sigma A)$ is a balanced dualizing*

complex (see [7, dfn. 3.7]*), and let M and N be A-bimodules which are finitely generated from either side.*

Suppose that A has the left AB *property. Then*

$$\mathrm{Ext}_A^{\gg 0}(M, N) = 0 \iff \mathrm{Ext}_A^{\gg 0}(N, {}^{\sigma}M) = 0.$$

Moreover, if A has finite Cohen-Macaulay type, then it has the left AB *property.*

Proof. The proof is once more an application of theorem 1.9.

The algebra A and the automorphism σ give the situation of setup 1.5 by proposition 3.9.

The balanced dualizing complex $\Sigma^d({}^{\sigma}A)$ has finite injective dimension from either side, so the same holds for ${}^{\sigma}A$. As an A-right-module, ${}^{\sigma}A$ is just A, so this implies $\mathrm{id}_{A^{\mathrm{op}}}(A) < \infty$. As an A-bimodule, ${}^{\sigma}A$ is $A^{\sigma^{-1}}$, and as an A-left-module, this is A, so it also implies $\mathrm{id}_A(A) < \infty$.

Since A has the left AB property, N and ${}^{\sigma}M$ viewed as A-left-modules have property (R) by proposition 2.4.

The A-bimodules N and ${}^{\sigma}M$ have property (S) by proposition 3.9.

Hence theorem 1.9 establishes the implications of the theorem, just as in the proof of theorem 4.3.

Finally, the balanced dualizing complex $\Sigma^d({}^{\sigma}A) \cong \Sigma^d(A^{\sigma^{-1}})$ is also pre-balanced by [7, lem. 3.3], so [7, prop. 3.4] gives

$$\mathrm{R}\Gamma_{\mathfrak{m}}(N)^L \cong \mathrm{RHom}_A(N, \Sigma^d(A^{\sigma^{-1}}))$$

when N is a finitely generated A-left-module. The $(-i)$'th cohomology of this is

$$\mathrm{R}^i\,\Gamma_{\mathfrak{m}}(N)^L \cong \mathrm{Ext}_A^{d-i}(N, A^{\sigma^{-1}}) \cong \mathrm{Ext}_A^{d-i}(N, A)^{\sigma^{-1}}.$$

This formula makes it clear that

$$N \text{ is maximal Cohen-Macaulay} \iff \mathrm{Ext}_A^{\geqslant 1}(N, A) = 0,$$

and so, if A has finite Cohen-Macaulay type, proposition 2.5 with $\mathcal{C}$ equal to a set of representatives of the isomorphism classes of indecomposable maximal Cohen-Macaulay modules implies that A has the left AB property. $\qquad\square$

Remark 4.7. There is a supply of algebras to which theorem 4.6 applies. Namely, consider the non-commutative special quotient surface singularities introduced in [6]. These are fixed point algebras of the form $A = B^G$ where G is a finite group acting suitably on the (non-commutative) complete local noetherian regular algebra B; see [6, sec. 3].

Such an A is complete local noetherian. It follows from [6, cor. 6.11, dfn. 6.15, and prop. 9.8] that $\Sigma^2(^\sigma A)$ is a balanced dualizing complex for A for some automorphism σ. And A has finite Cohen-Macaulay type by [6, thm. 9.10], so theorem 4.6 applies.

In another direction, [11, cor. 2.4] says that if A is a noetherian ring with finite global dimension, then $A/(x_1, \ldots, x_n)$ has the left AB property when $x_1, \ldots, x_n$ is a regular sequence of normal elements where conjugation by x_{i+1} has finite order on $A/(x_1, \ldots, x_i)$ for each i. This gives another way of getting algebras to which theorems 4.3 and 4.6 apply.

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